Application of Differential Synthesis to Design of Multiaxis Stability Augmentation Systems

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A differential synthesis technique is developed for multiple input linear feedback systems that allows direct calculation of the set of feedback gains and control interconnects that yield arbitrarily selected flying qualities parameters. This is accomplished by parametrically relating differential gain and interconnect changes to differential changes in flying qualities parameters. The procedure is illustrated by synthesizing a linear feedback control for the linearized lateral dynamics of a lifting-body entry vehicle with unacceptable flying qualities in the unaugmented condition. The differential synthesis method developed in this paper is suitable for digital computer formulation and is easily adapted to systems of high order.

Nomenclature

A = differential transition matrix
A_ϕ = leading coefficient of the numerator quadratic in the ϕ/δ_a transfer function
$B(s)$ = matrix polynomial in s
B_k = matrix coefficients of powers of s in $B(s)$
c = vector containing the d_k and α_{ijk} coefficients
C = input matrix
C_{l_β} = rolling moment coefficient due to sideslip
C_{l_p} = damping in roll coefficient
C_{l_r} = rolling moment coefficient due to yaw rate
$C_{l_{\delta_a}}$ = aileron rolling moment coefficient
$C_{l_{\delta_r}}$ = rudder rolling moment coefficient
C_{n_β} = static directional stability coefficient
C_{n_p} = yawing moment coefficient due to roll rate
C_{n_r} = damping in yaw coefficient
$C_{n_{\delta_a}}$ = aileron yawing moment coefficient
$C_{n_{\delta_r}}$ = rudder yawing moment coefficient
C_{y_β} = lateral force coefficient due to sideslip
C_{y_p} = lateral force coefficient due to rolling velocity
C_{y_r} = lateral force coefficient due to yawing velocity
$C_{y_{\delta_a}}$ = lateral force coefficient due to aileron deflection
$C_{y_{\delta_r}}$ = lateral force coefficient due to rudder deflection
$d(s)$ = characteristic polynomial of A
d_k = coefficients of powers of s in $d(s)$
$f(g)$ = vector function of vector g, $f(g) = c$
F = control interconnect matrix
g = vector containing elements of the gain and interconnect matrices
G = feedback gain matrix
$\text{grad}_g f$ = matrix whose element in ith row and jth column is $\partial f_i/\partial g_j$
$\text{grad}_c P$ = matrix whose element in ith row and jth column is $\partial P_i/\partial c_j$
$\text{grad}_q P$ = matrix whose element in ith row and jth column is $\partial P_i/\partial q_j$
H = augmented differential transition matrix
I = commensurable identity matrix
I_x = rolling moment of inertia
I_z = yawing moment of inertia
I_{xz} = product of inertia
K = augmented input matrix
m = dimension of u
n = dimension of x
p = roll rate
$P(c,q)$ = algebraic identities between c and q
q = vector of flying quality parameters
r = yaw rate
s = Laplace transform variable
u = input control vector
u_p = input control vector to the augmented system
W = weighting matrix
x = state vector
α_{ijk} = coefficient of the $(n-k)$th power of s in the x_i/u_j transfer function numerator
β = sideslip angle, deg
δ_a = aileron deflection, deg
δ_r = rudder deflection, deg
μ_k = commensurable unit vector with 0 in all but kth position
ω_ϕ = natural frequency of numerator quadratic in the ϕ/δ_a transfer function
ω_d = natural frequency of dutch roll mode
ϕ = bank angle, deg
σ = scalar used to parameterize gain variations
τ_s = spiral mode time constant
τ_r = roll mode time constant
ζ_ϕ = damping ratio of numerator quadratic in the ϕ/δ_a transfer function
ζ_d = damping ratio of the dutch roll mode

Subscripts

i,j,k,p,q = scalar components of a vector (used singly) or elements of a matrix (used doubly)

Superscripts

t = transpose of a vector or matrix
0 = function evaluated at $\sigma = 0$
1 = function evaluated at $\sigma = 1$

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Introduction

HIGH-performance aircraft must be provided with good flying qualities over the design flight envelope. When this requirement cannot be met with a given configuration, automatic control is introduced to alter the stability and control characteristics of the aircraft so that the flying qualities are improved. In terms of linear system theory, automatic control systems enable alteration of the aircraft stability and control characteristics by using feedback and control interconnect networks that change the pole-zero configuration of aircraft response transfer functions. One problem in the design of linear control systems is the synthesis of the feedback gains and interconnect ratios required to establish a desirable pole-zero configuration of the aircraft response transfer functions.

Reference 1 has considered determination of the aircraft lateral control system gains required to establish a specified pole configuration. Reference 2 considers the design problem for multiple input control systems from the standpoint of optimal control theory but does not provide a general method for establishing specified pole-zero requirements.

The purpose of this paper is to develop and demonstrate a direct method for determining gains and interconnect ratios required to establish a specific pole-zero configuration for a general multiple-input linear system. The method developed combines linear systems theory and differential calculus and is applicable to systems of high order. This method is called differential synthesis. After the method is formulated, it is applied to the determination of the gains and control interconnects required for establishment of various pole-zero configurations of the lateral response transfer functions for a lifting-body entry vehicle.

Analytical Development

Basic System Theory

The basic linearized equations of motion of aircraft, using state vector notation, take the form of a multi-input system

$$\dot{x} = Ax + Cu \tag{1}$$

where $x = n$-dimensional state vector, $u = m$-dimensional control vector, $A = n \times n$ differential transition matrix, and $C = n \times m$ input matrix.

The Laplace transform of Eq. (1) is

$$x(s) = (sI - A)^{-1}Cu(s) \tag{2}$$

and $(sI - A)^{-1}$ is of the form $B(s)/d(s)$, where

$$\left.\begin{aligned} B(s) &= B_1 s^{n-1} + B_2 s^{n-2} + \ldots + B_n \\ d(s) &= s^n + d_1 s^{n-1} + d_2 s^{n-2} + \ldots + d_n \end{aligned}\right\} \tag{3}$$

Leverrier's algorithm for obtaining the B_k matrices and the d_k coefficients appearing in Eq. (3) is presented in Ref. 3 and below

$$\left.\begin{aligned} B_1 &= I & d_1 &= -\text{trace}(B_1 A) \\ B_2 &= B_1 A + d_1 I & d_2 &= -\tfrac{1}{2}\text{trace}(B_2 A) \\ &\vdots & &\vdots \\ B_k &= B_{k-1} A + d_{k-1} I & d_k &= -\frac{1}{k}\text{trace}(B_k A) \\ &\vdots & &\vdots \\ B_n &= B_{n-1} A + d_{n-1} I & d_n &= -\frac{1}{n}\text{trace}(B_n A) \end{aligned}\right\} \tag{4}$$

The transfer function between the ith component of x and the jth component of u from Eq. (2) is of the form

$$\frac{x_i}{u_j} = \frac{\alpha_{ij1}s^{n-1} + \alpha_{ij2}s^{n-2} + \ldots + \alpha_{ijn}}{s^n + d_1 s^{n-1} + d_2 s^{n-2} + \ldots + d_n} \tag{5}$$

where the d_k coefficients can be obtained from Eq. (4) and the coefficients α_{ijk} can be computed from the expression

$$\alpha_{ijk} = \mu_i{}^t B_k C \mu_j \tag{6}$$

where μ_i and μ_j are commensurable unit vectors with 0 in all but the ith and jth positions, respectively. The pole-zero configuration of the transfer function [Eq. (5)] is implicitly determined by the coefficients α_{ijk} and d_k.

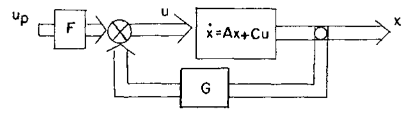

Fig. 1 System block diagram.

Statement of the Problem

In order to alter the transfer function coefficients and thus, the pole-zero configuration a feedback control system is introduced as schematically illustrated in Fig. 1. The system has the form

$$u = Gx + Fu_p \tag{7}$$

where $F = m \times m$ control interconnect matrix, $G = m \times n$ linear feedback gain matrix, and $u_p = m$-dimensional input vector.

When Eq. (7) is substituted into Eq. (1), the system becomes

$$\dot{x} = Hx + Ku_p \tag{8}$$

where

$$\left.\begin{aligned} H &\triangleq A + CG \\ K &\triangleq CF \end{aligned}\right\} \tag{9}$$

The system transfer function becomes

$$x(s) = (sI - H)^{-1}Ku_p(s) \tag{10}$$

The transfer function between the x_i component of x and the u_{pj} component of u_p has the same form as Eq. (5) where the α_{ijk} and d_k coefficients have been obtained by substitution of H for A in Eq. (4) and K for C in Eq. (6).

The problem considered here is to determine a gain matrix G and an interconnect matrix F that result in a specific set of coefficients α_{ijk} and d_k corresponding to the transfer function of interest x_i/u_j.

Method of Solution

One way to determine the matrices G and F is, after substituting Eq. (9) into Eqs. (4) and (6), to expand the Eqs. (4) and (6) in terms of the elements of G and F and construct a set of nonlinear algebraic equations of the form

$$f(g) = c \tag{11}$$

Here the transfer function coefficients c are

$$c^t = (d_1, \ldots, d_n; \quad \alpha_{ij1}, \ldots, \alpha_{ijn}) \tag{12}$$

the vector of independent variables g is

$$g^t = (g_{11}, \ldots, g_{1n}, \ldots, g_{mn}; \quad f_{11}, \ldots, f_{1m}, \ldots, f_{mm}) \tag{13}$$

and $f(g)$ is the set of nonlinear functions representing the relationship between the elements of g and c. Equation (11) must then be solved for a set of gain and interconnect matrix elements that result in a specified $c = c^1$.

For a complex set of equations such as those representing aircraft dynamics, Eq. (11) is difficult to construct and not generally easy to solve. In order to avoid these difficulties, the method of solution proposed here is to convert the nonlinear algebraic set of equations into a set of implicit differential equations with known initial conditions. After developing this method for solving nonlinear algebraic equations, the authors discovered that M. N. Yakovlev had suggested this procedure in 1965 (Ref. 4).

In order to best explain the method of conversion to differential form for solving nonlinear algebraic equations consider the scalar equation

$$f(g) = g^2 - 3g + 2 = c \tag{14}$$

The graph of g vs c satisfying Eq. (14) is presented in Fig. 2. Assume that the object is to determine a value of g that satisfy $f(g) = 0$. First, let both g and c be functions of a

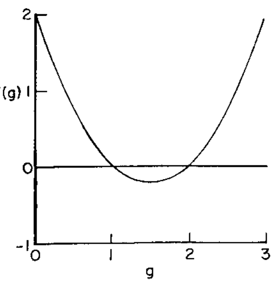

Fig. 2 Graph of the scalar function defined by Eq. (14).

dummy variable σ which varies between 0 and 1. That is, $c = c(\sigma)$ and $g = g(\sigma)$. The basic idea is to choose an arbitrary $g^0 = g(0)$ and select a variation $c(\sigma)$ such that it passes through the point $c(0) = f(g^0)$ and the desired point $c(1) = 0$. A typical curve $c(\sigma)$ satisfying these requirements is the straight line illustrated in Fig. 3. Here $c(0)$ was calculated by substituting $g = g^0 = 0$ into Eq. (14). Then, the variation in $g(\sigma)$ required by the identity $f[g(\sigma)] \equiv c(\sigma)$ is determined by using a differential form of Eq. (14), which is $\{\partial f[g(\sigma)]/\partial g\}[dg(\sigma)/d\sigma] = dc(\sigma)/d\sigma$. For this particular example using the straight-line variation indicated in Fig. 3 for $c(\sigma)$ the derivative $dg(\sigma)/d\sigma$ must satisfy

$$dg(\sigma)/d\sigma = -2/(2g - 3) \tag{15}$$

This equation may be integrated numerically over the interval $0 \leq \sigma \leq 1$ using the chosen initial condition $g(0) = 0$. The value of g at $\sigma = 1$ satisfies $f[g(1)] = c(1) = 0$ and a solution to the equation $f(g) = 0$ is obtained.

It is obvious that for the simple example above, it is easier to solve the nonlinear algebraic equation using the quadratic formula than to apply a conversion to differential form. However, for complex linear systems a conversion to differential form has advantages which will be discussed.

Conversion to differential form can be applied to the synthesis of linear systems by solving Eq. (11) in a manner similar to that used for the scalar Eq. (14). Assume that the object is to select a vector c^1 consisting of desired transfer function coefficients and determine a gain and interconnect vector g^1 that satisfies $f(g^1) = c^1$. Let c and g be functions of the dummy variable σ and select $c(\sigma)$ such that it satisfies $c(0) = c^0$ where

$$c^0 \triangleq f(g^0)$$

$c(1) = c^1$, and is differentiable on the internal $0 \leq \sigma \leq 1$. The linear function

$$c(\sigma) = c^0 + \sigma(c^1 - c^0) \tag{16}$$

is an example. The variation in $g(\sigma)$ required by the identity $f[g(\sigma)] = c(\sigma)$ must satisfy the equation

$$\{\text{grad}_g f[g(\sigma)]\}[dg(\sigma)/d\sigma] = dc(\sigma)/d\sigma \tag{17}$$

which is Eq. (11) converted to differential form. Equation (17) is a set of implicit differential equations which are linear in the derivatives $dg/d\sigma$ and which can usually be integrated numerically over the interval $0 \leq \sigma \leq 1$ using the initial conditions $g(0) = g^0$. The right side of the equation (i.e., $dc/d\sigma$) can be thought of as a *forcing function* requiring $g(\sigma)$ to follow some path which preserves the identity $f[g(\sigma)] = c(\sigma)$. Therefore, the value of $g(\sigma)$ at $\sigma = 1$ should satisfy the equation $f[g(1)] = c^1$ and a gain and interconnect vector g^1 is obtained that results in a c^1 vector of desired transfer function coefficients. During integration of Eq. (17), singularities may arise due to the implicit nature of these differential equations. A discussion of these singular situations is presented in a following section.

An advantage of this synthesis technique is that the vector c and the matrix $\text{grad}_g f[g(\sigma)]$ can be easily formulated using Leverrier's algorithm and a differentiated form of this algorithm. Thus, the complicated nonlinear algebraic functions $f(g)$ need not be expanded into scalar form and Eq. (17) can be constructed as well as solved using a digital computer. The method of calculating $\text{grad}_g f[g(\sigma)]$ is presented in a following section.

Forcing Functions

In order to obtain a suitable forcing function $dc(\sigma)/d\sigma$, the function $c(\sigma)$ must be determined. As previously stated $c(\sigma)$ must pass through the initial and desired vectors of transfer function coefficients. The vector of initial coefficients is calculated by assuming some initial gain vector g^0 which is made up of the elements of the gain matrix G and the interconnect matrix F. Usually, suitable initial conditions are $G = 0$ and $F = I$ which is equivalent to the open-loop system without control augmentation. However, in some cases when the off-diagonal elements of F are to be changed, their initial values should be nonzero to avoid computational difficulties. The selection of desirable coefficients can be accomplished for the airplane from established flying qualities parameters. These parameters are determined by the desired transfer function pole-zero locations. For other applications similar criteria would have to be obtained to establish a desired value c^1.

The first function $c(\sigma)$ to be used was the linear variation of Eq. (16). It was found that with the resulting forcing function

$$dc(\sigma)/d\sigma = c^1 - c^0 \tag{18}$$

integration errors which occur during integration of Eq. (17) were not reduced but remained uncorrected. However, the effect of integration errors is reduced by modifying Eq. (18) so that the slope $dc(\sigma)/d\sigma$ is continuously updated. The modified linear forcing function that accomplishes this updating is

$$dc/d\sigma = [c^1 - f(g)]/[1 - \sigma] \tag{19}$$

This function has been used successfully by the authors in many applications. This closed-loop forcing function greatly improves the computational accuracy of obtaining the desired c^1 and thus g^1. Equations (18) and (19) are the only two forcing functions which have been used by the authors; however, others may be satisfactory.

In some applications, it is desirable to determine the gain vector g required to specify functions of the transfer function polynomial coefficients instead of the coefficients directly. These functions can be the parameters obtained when the polynomials are factored. To do this requires a modification of the forcing function as well as the gradient matrix. Consider then the problem of specifying a set of parameters q that are algebraically related to the elements d_k and α_{ijk} of the vector c by the equation

$$P(c,q) = 0 \tag{20}$$

Cases 1 and 3 in the section on applications are examples in which this problem arises. Using the same concept developed to specify c, Eq. (20) is formally differentiated to yield

$$\text{grad}_q P(dq/d\sigma) + \text{grad}_c P(dc/d\sigma) = 0$$

Hence, from Eq. (17) a relation between the variations in q and the gain and interconnect matrix variations is established

$$\text{grad}_c P \ \text{grad}_g f(dg/d\sigma) = -\text{grad}_q P(dq/d\sigma) \tag{21}$$

In Eq. (21), the quantity $dq/d\sigma$ on the right side acts as the forcing term and is selected arbitrarily. Equation (21) is then numerically integrated to obtain the vector $g(\sigma)$ required to obtain the $q(\sigma)$ variation selected. The application of Eq. (21) follows along the same line as that of Eq. (17).

Gradient Matrix

To construct the matrix $\text{grad}_g f(g)$, first consider the partial derivatives of $f(g)$ with respect to the gain matrix elements. Let dG represent an infinitesimal change in the gain matrix G. This change causes infinitesimal changes in the B_k matrices and the d_k coefficients of Eqs. (4) when H is substituted for A. These matrices and coefficients obey the equations

$$\begin{aligned} dB_1 &= 0 \qquad d(d_1) = -\text{trace}(CdG) \\ dB_k &= (dB_{k-1})H + B_{k-1}CdG + Id(d_{k-1}) \\ d(d_k) &= -(1/k)\,\text{trace}[(dB_k)H + B_kCdG] \\ k &= 2,3,\ldots,n \end{aligned} \tag{22}$$

where dB_k and $d(d_k)$ are infinitesimal changes in the B_k matrices and d_k coefficients. A differential form of Leverrier's algorithm has previously been used by Morgan (Ref. 5) to study sensitivity. Equation (22) is linear in the matrix dG so that by setting all the dG elements to zero except dG_{pq} which is set to unity ($dG = \mu_p\mu_q^t$), the result is that $d(d_k) = \partial d_k/\partial g_{pq}$ and $dB_k = \partial B_k/\partial g_{pq}$. The partial derivatives $\partial d_k/\partial g_{pq}$ are elements of one column of $\text{grad}_g f(g)$. To obtain the $\partial\alpha_{ijk}/\partial g_{pq}$ elements of the same column the partial derivatives $\partial B_k/\partial g_{pq}$ are used in a differential form of Eq. (6),

$$(\partial\alpha_{ijk}/\partial g_{pq}) = \mu_i^t(\partial B_k/\partial g_{pq})K\mu_j \tag{23}$$

Thus, by repeated substitution, the gradient matrix elements related to G can be obtained.

Now consider elements of $\text{grad}_g f(g)$ which are partial derivatives of $f(g)$ with respect to the interconnect matrix elements f_{pq}. Note that the coefficients d_k and the matrices B_k are independent of the matrix K so that $\partial d_k/\partial f_{pq} = 0$ and $\partial B_k/\partial f_{pq} = 0$. Hence, from Eqs. (6) and (9)

$$\partial\alpha_{ijk}/\partial f_{pq} = \mu_i^t(\partial B_k/\partial f_{pq})K\mu_j^t + \mu_i B_k C(\partial F/\partial f_{pq})\mu_j = \mu_i^t B_k C\mu_p\mu_q^t\mu_j$$

using $\partial F/\partial f_{pq} = \mu_p\mu_q^t$. Thus,

$$\frac{\partial\alpha_{ijk}}{\partial f_{pq}} = \begin{cases} \mu_i^t B_k C\mu_p & j = q \\ 0 & j \neq q \end{cases} \tag{24}$$

Again by repeated substitution, the gradient matrix columns related to F can be obtained.

Singularities

During the integration of Eq. (17), there may be instances in which the determinant of $\text{grad}_g f[g(\sigma)]$ goes to zero. At these singular situations, there is a restricted set of variations in $c(\sigma)$ that can be realized by variations in $g(\sigma)$. That is, the variations in $c(\sigma)$ are not completely arbitrary and in some instances the desired c^1 may not be obtainable. A thorough investigation of these singular situations has not been undertaken; however, it is known that they arise when $g(\sigma)$ passes through a local maxima or minima of the elements of $f(g)$ or when the poles or zeros of the transfer functions being altered changes from pairs of complex roots to pairs of real roots and vice versa. Two techniques have been used to avoid or bypass these singularities. These are applicable only when the number of transfer function coefficients being varied is less than the total number of gain and interconnect matrix elements available. In order to explain these techniques, first assume that only feedback gains can be altered. Further assume that there are eight gains available and only six transfer function coefficients are to be altered. Conceptually, only six gains should be required to alter six coefficients, but in order to avoid singularities, seven gains are allowed to vary.

Table 1 Vehicle configuration and aerodynamic data[a]

I_x =	1710 slug − ft^2	Weight =	10,000 lbs		
I_z =	7600 slug − ft^2	Wing span =	15.25 ft		
I_{xz} =	500 slug − ft^2	Wing area =	180 ft^2		
C_{l_p} =	−0.15	C_{n_p} =	0	C_{y_p} =	0
C_{l_r} =	0	C_{n_r} =	−0.2	C_{y_r} =	0
C_{l_β} =	−0.05	C_{n_β} =	0.05	C_{y_β} =	−0.8
$C_{l_{\delta_a}}$ =	−0.015	$C_{n_{\delta_a}}$ =	0.02	$C_{y_{\delta_a}}$ =	0
$C_{l_{\delta_r}}$ =	0.005	$C_{n_{\delta_r}}$ =	−0.02	$C_{y_{\delta_r}}$ =	0.04

[a] Body axis aerodynamic coefficients are in rad.

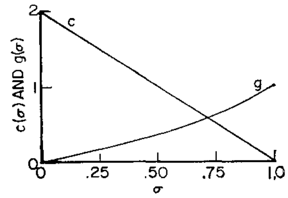

Fig. 3 Graph of the $c(\sigma)$ variation selected and the $g(\sigma)$ variation which results from the solution of Eq. (15).

The first technique used to avoid singularities is based on minor determinants of $\text{grad}_g f(g)$. During each integration step, any six of the seven chosen gains are allowed to vary with the seventh remaining constant at its last value. The combination of six gains can change from step to step and is chosen at the beginning of each step by determining which of the seven possible six by six gradient matrices has the largest determinant. Using this procedure gain combinations which result in gradients with determinants near zero are avoided and thus, singularities are avoided. This technique was used for all the examples presented in the application section.

The second technique used to avoid singularities utilizes optimum methods to calculate $dg(\sigma)/d\sigma$. The performance index $[dg^t(\sigma)/d\sigma](W/2)[dg(\sigma)/d\sigma]$ is minimized subject to the constraint of Eq. (17). Applying basic optimization techniques

$$dg/d\sigma = W^{-1}\,\text{grad}_g{}^t f(g)[\text{grad}_g f(g)W^{-1}\,\text{grad}_g{}^t f(g)]^{-1}dc/d\sigma$$

Here, all seven of the chosen gains vary during each integration step and the weighting matrix W can be used to place more or less emphasis on a particular gain. This technique has been used successfully in all applications attempted and requires significantly less computer time than the first technique.

It should be noted that the solutions for g obtained by this synthesis technique are not unique. This is illustrated by the example of Fig. 2. The result of integrating Eq. (15) with the initial condition $g(0) = 0$ leads to the solution $g(1) = 1$. However, the initial condition $g(0) = 3$ is also appropriate for Eq. (15) but it leads to the result $g(1) = 2$. The two techniques used to bypass singularities also have nonunique results. In these cases, however, the solution obtained represents only one of a complete family of possible solutions.

Applications

The differential synthesis technique developed in the preceding section was applied to determining feedback gains and control interconnect ratios to obtain various pole-zero configurations for the transfer functions related to the lateral response of a lifting-body entry vehicle. A flight condition at a Mach number of 1.8, an altitude of 60,000 ft, and an angle of attack of 15° was used in this study. Table 1 contains the vehicle configuration and aerodynamic data used. The linearized aircraft lateral equations of motion used can be found in Ref. 6 and were formulated according to Eq. (1) with $x^t \triangleq (x_1,x_2,x_3,x_4) = (p,\phi,r,\beta)$ and $u^t \triangleq (u_1,u_2) = (\delta_a,\delta_r)$.

One of the important indicators of the quality of lateral response is the bank angle to aileron transfer function. Flight and simulator experience has shown that seven flying quality parameters in this transfer function influence pilot opinion

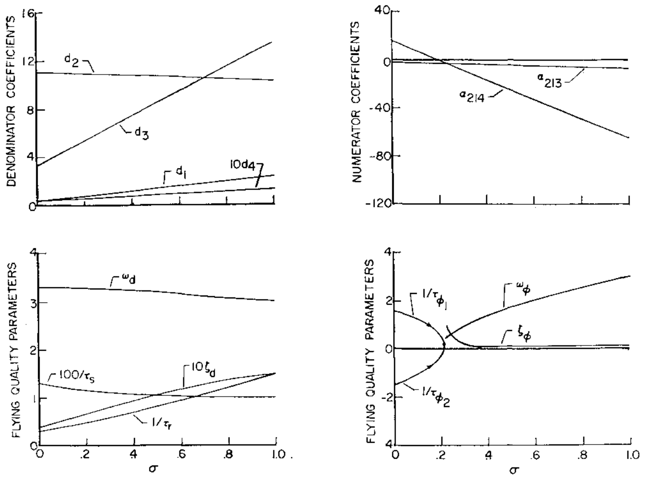

Fig. 4 Variation of ϕ/δ_a transfer function coefficients and flying quality parameters with σ.

of lateral response. These are A_ϕ, ξ_ϕ, ω_ϕ, τ_s, τ_r, ζ_d, and ω_d which appear in the bank angle to aileron transfer function

$$\frac{\phi}{\delta_a} = \frac{A_\phi(s^2 + 2\zeta_\phi\omega_\phi s + \omega_\phi^2)}{[s + (1/\tau_s)][s + (1/\tau_r)](s^2 + 2\zeta_d\omega_d s + \omega_d^2)} \quad (25)$$

When written in the form of Eq. (5), this equation is

$$\frac{\phi}{\delta_a} = \frac{x_2}{u_1} = \frac{\alpha_{212}s^2 + \alpha_{213}s + \alpha_{214}}{s^4 + d_1s^3 + d_2s^2 + d_3s + d_4} \quad (26)$$

If the flying quality parameters are unsatisfactory, the coefficients d_k and α_{ijk} can be adjusted so that desired flying quality parameters are obtained. Desirable flying quality parameters for the ϕ/δ_a transfer function were based on information found in Refs. 7 and 8 which indicate that pole-zero cancellation is desirable in order to obtain good response. Numerical values of the desired flying quality parameters used in the study as well as those of the basic vehicle are listed in Table 2. The lateral response of the basic vehicle is unacceptable at this flight condition because of roll reversal indicated by the negative value of ω_ϕ^2. Also, the value of $1/\tau_r$ indicates the roll subsidence damping is too low and the value of ζ_d indicates the dutch roll mode damping is too low. This basic case has been selected since it demonstrates some design problems that may be encountered.

Illustrative Example

An example case is presented in detail to illustrate the mechanics of the application of the synthesis technique presented. For this example, the objective was to determine a gain matrix G required to obtain particular values of all the numerator and denominator coefficients except α_{212} of Eq. (26) using a control interconnect matrix of $F = I$. Thus, six coefficients were changed and the vector c was of the form $c^t = (d_1,d_2,d_3,d_4,\alpha_{213},\alpha_{214})$. Since only six coefficients were to be changed, in concept only six gain matrix elements should have been required. However, to avoid singularities in the gradient matrix of Eq. (17), seven feedback gains were employed. Hence, for this example all gain matrix elements were used except δ_r/ϕ and Eq. (17) was implemented with $g^t = (g_{11},g_{12},g_{13},g_{14},g_{21},g_{23},g_{24}) = (\delta_a/p,\delta_a/\phi,\delta_a/r,\delta_a/\beta,\delta_r/p,\delta_r/r,\delta_r/\beta)$.

For this example, the initial conditions at $\sigma = 0$ corresponded to the unaugmented aircraft with $G = 0$ and $F = I$. The closed-loop form of the forcing function on the right side of Eq. (17) $dc/d\sigma = [c^1 - f(g)]/(1 - \sigma)$ was used with c^1 taken as the list of desired coefficients contained in Table 2. Equation (17) was then numerically integrated from $\sigma = 0$ to $\sigma = 1$ in steps of 0.1 using a fourth-order Runge-Kutta integration process. The variation of the coefficients (components of c) with σ is illustrated in Fig. 4. This shows that the selected linear variation in $c(\sigma)$ with σ was followed very closely by $f(g)$ and that the components of c at $\sigma = 1$ were equal to the components of c^1. Figure 4 also illustrates the variation of the associated flying quality parameters with σ that results from the linear variation of the coefficients. Note that at $\sigma = 0$ the numerator polynomial has two real zeros $1/\tau_{\phi_1}$ and $1/\tau_{\phi_2}$. As σ increases the real zeros meet and then break away into a pair of complex zeros at $\sigma = 0.22$.

The gain variations that were calculated to establish the coefficient variations of Fig. 4 are shown in Fig. 5. From $\sigma = 0$ to $\sigma = 0.2$, six gains were varied and the δ_a/r gain was held constant at zero. In the neighborhood of $\sigma = 0.22$, a singularity occurred in the gradient matrix because of the change in analytic form of the numerator polynomial and the integration process could not continue holding $\delta_a/r = 0$. However, this singularity was avoided and the integration step was completed by holding the δ_r/p gain constant and allowing the remaining six gains to vary. The numerical integration process may be slightly inaccurate in the neighborhood of this singularity; however, due to the closed-loop programing of the coefficient variations this inaccuracy has little effect on the outcome of the integration process at $\sigma = 1$. From $\sigma = 0.3$ to $\sigma = 0.7$, the δ_a/r gain was again held constant. Another type of singularity occurred in the neigh-

Table 2 Coefficients and flying quality parameters for the ϕ/δ_a transfer function

	Basic	Desired
	Coefficients	
d_1	0.57	2.41
d_2	11.17	10.37
d_3	3.43	13.60
d_4	0.043	0.14
α_{212}	−7.14	[a]
α_{213}	−1.32	−6.43
α_{214}	17.89	−64.30
	Flying quality parameters	
$1/\tau_s$	0.013	0.01
$1/\tau_r$	0.296	1.5
ζ_d	0.039	0.15
ω_d	3.33	3.0
A_ϕ	−7.14	[a]
ζ_ϕ	[b]	0.15
ω_ϕ	[b]	3.0
ω_ϕ^2/ω_d^2	−0.226	1.0

[a] Not specified.
[b] For the basic case, the numerator quadratic in the ϕ/δ_a transfer function has two real zeros: $1/\tau_{\phi_1} = 1.68$ and $1/\tau_{\phi_2} = -1.5$.

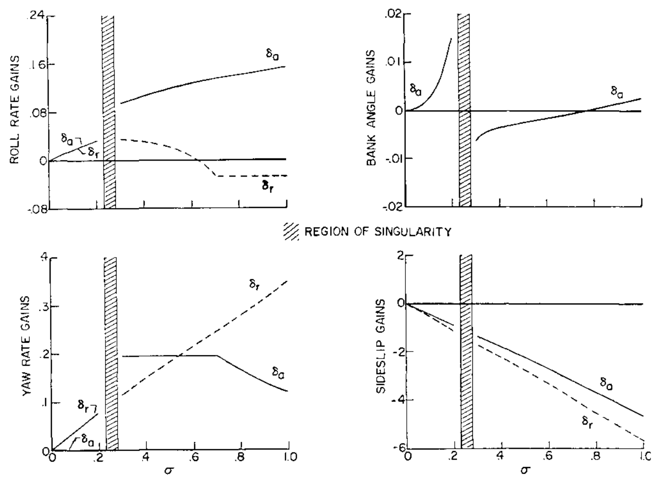

Fig. 5 Variation of feedback gains with σ.

Table 3 Feedback gain and control interconnect matrices that yield pole-zero configurations selected

	F	G
	$F = \begin{bmatrix} \delta a/\delta a_p & \delta a/\delta r_p \\ \delta r/\delta a_p & \delta r/\delta r_p \end{bmatrix}$	$G = \begin{bmatrix} \delta a/p & \delta a/\phi & \delta a/r & \delta a/\beta \\ \delta r/p & \delta r/\phi & \delta a/r & \delta r/\beta \end{bmatrix}$
Case 1	$F = \begin{bmatrix} 1 & 0 \\ 1.24 & 1 \end{bmatrix}$	$G = \begin{bmatrix} 0.392 & 0 & 0.746 & 0 \\ 0.296 & 0 & 0.392 & 0 \end{bmatrix}$
Case 2	$F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$G = \begin{bmatrix} 0.153 & 0.002 & 0.118 & -4.64 \\ -0.028 & 0 & 0.348 & -5.69 \end{bmatrix}$
Case 3	$F = \begin{bmatrix} 1 & 0 \\ 1.39 & 1 \end{bmatrix}$	$G = \begin{bmatrix} 0.284 & -0.004 & -0.149 & -0.537 \\ 0.337 & 0 & 0.027 & 0 \end{bmatrix}$
Case 4	$F = \begin{bmatrix} 1 & 0.201 \\ 1.39 & 1 \end{bmatrix}$	$G = \begin{bmatrix} 0.205 & 0.002 & 0.135 & -4.64 \\ 0.230 & -0.003 & 0.430 & -5.69 \end{bmatrix}$

borhood of $\sigma = 0.7$. This is evidenced by the rapidly decreasing δ_r/p gain. By holding δ_r/p constant and allowing the remaining six gains to vary, the integration process was continued to $\sigma = 1$. At $\sigma = 1$, the gains that yield the desired numerator and denominator coefficients of Eq. (26) were obtained. Although in this case seven gains were used to change six coefficients, in some cases the additional gain may not be necessary. A Control Data Corporation (CDC) 6000 series digital computer was used to obtain the results for this case and the run time was 8 sec.

Flying Quality Considerations

A short analytical flying qualities study is presented here to illustrate the variety of design objectives that can be established using the synthesis technique presented. Four different sets of design objectives were used to obtain four control systems that would improve the vehicle lateral response characteristics. The lateral response of the basic vehicle to a 5° step aileron input is shown in Fig. 6. This figure illustrates the problems of roll reversal, induced sideslip, and underdamped roll subsidence and dutch roll modes.

For case 1, the design objective was to modify the stability and control characteristics so that the desired values of $1/\tau_s$, $1/\tau_r$, ζ_d, ω_d, and $\omega_\phi{}^2$ which are listed in Table 2 were obtained. This was accomplished by changing the d_1, d_2, d_3, d_4, α_{212}, and ρ_{214} coefficients. The feedback gains δ_a/p, δ_a/r, δ_r/p, and δ_r/r and the interconnect δ_r/δ_{a_p} were used. The set of feedback gains used were selected since rate feedback is a common form of stability augmentation. The initial value of δ_r/δ_{a_p} at $\sigma = 0$ was taken as unity. This case illustrates specification of a function of the coefficients α_{ijk} as opposed to specifying the coefficients themselves. Here, the ratio $\alpha_{214}/\alpha_{212}$ was specified rather than α_{212} and α_{214} individually to obtain the desired $\omega_\phi{}^2$. The gain and interconnect matrices calculated to meet the design objective are presented in Table 3. The response of the augmented vehicle to a 5° step δ_{a_p} input is shown in Fig. 7. Although the roll reversal problem has been corrected, there is still a significant induced sideslip. Other control system designs were studied using different combinations of gain matrix elements. It was found that

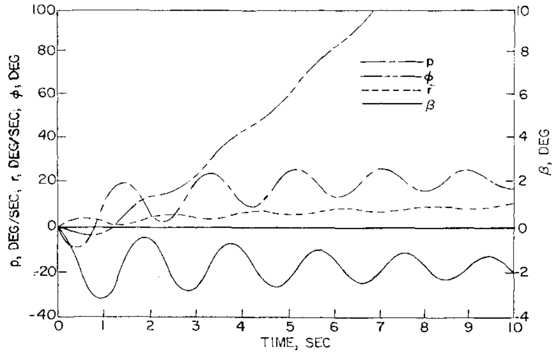

Fig. 6 Time history of the response of the basic vehicle to a 5° step aileron input.

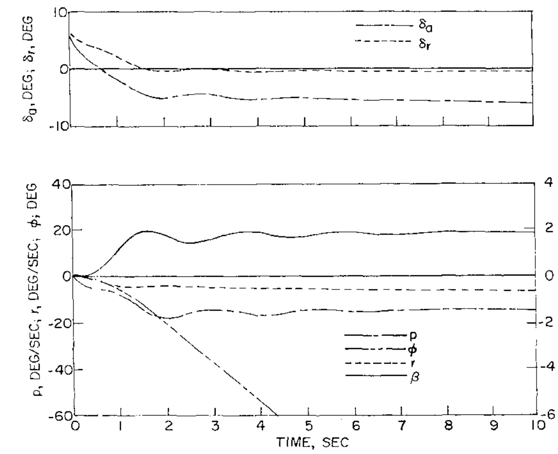

Fig. 7 Time history of the response of the augmented vehicle of case 1 to a 5° step aileron input.

the induced sideslip could be significantly reduced using the gain elements δ_a/p, δ_a/β, δ_r/p, δ_r/r.

Case 2 illustrates the design objective of obtaining pole-zero cancellation in the ϕ/δ_a transfer function using feedback only. This case was previously presented as the illustrative example. The gain matrix calculated to meet the design requirements is presented in Table 3 and the response of the augmented vehicle to a 5° step δ_{a_p} input is presented in Fig. 8. Figure 8 illustrates that the p and ϕ responses are good but there still remains an undesirable β transient. This indicates that the flying quality parameters specified in this example were not adequate for insuring turn coordination.

Since only the G matrix was used in case 2, case 3 was undertaken specifying the same flying quality parameters but including a control interconnect as a possible means of eliminating the β response shown in Fig. 8. Instead of specifying the values of the numerator coefficients, the functions $\alpha_{213}/\alpha_{212}$ and $\alpha_{214}/\alpha_{212}$ were specified to obtain the desired $2\zeta_\phi\omega_\phi$ and $\omega_\phi{}^2$, respectively. The gain and control interconnect matrices calculated to obtain the desired flying quality parameters are shown in Table 3 and a time history of the lateral response to a 5° step δ_{a_p} input is presented in Fig. 9. Note that the undesirable β transient has almost been eliminated from the response. Hence, β is almost decoupled from the δ_{a_p} inputs. This occurs since the resulting values of the numerator coefficients in the β/δ_{a_p} transfer function are very small.

In case 3, the β transient has almost been eliminated from the aileron input. It may also be possible to eliminate the bank angle transients from rudder inputs. This "decou-

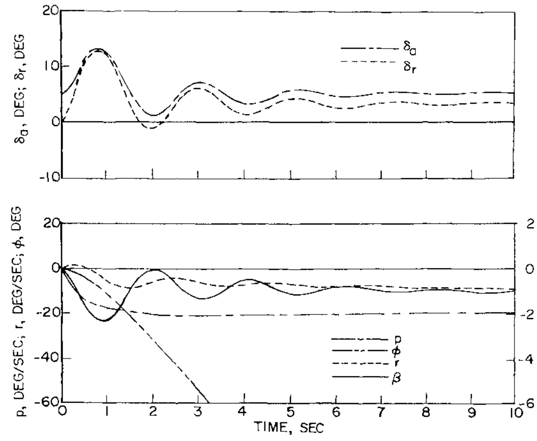

Fig. 8 Time history of the response of the augmented vehicle of case 2 to a 5° step aileron input.

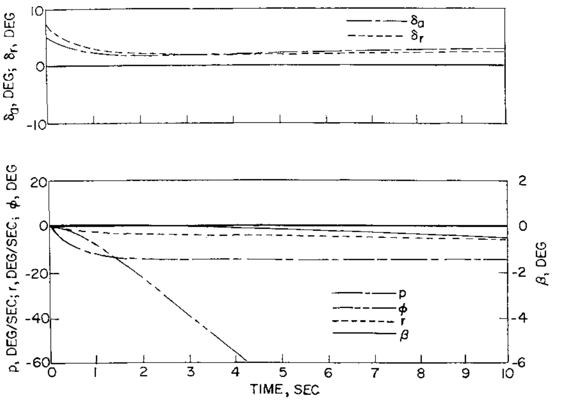

Fig. 9 Time history of the response of the augmented vehicle of case 3 to 5° step aileron input.

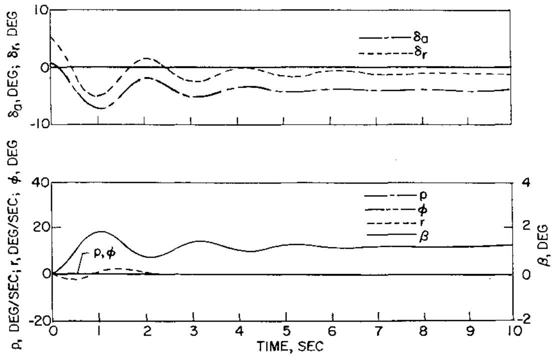

Fig. 11 Time history of the response of the augmented vehicle of case 4 to a 5° step rudder input.

pling" design has been the subject of study by several authors (e.g., Ref. 9). The synthesis technique presented herein can be applied to the decoupling problem by specifying that the numerators of the β/δ_{a_p} and ϕ/δ_{r_p} transfer functions vanish. However, to accomplish this and still specify vehicle stability characteristics more gain and control interconnect matrix elements are required than are available. It was noted that if $C_{y\delta_r}$ were neglected, enough gain and control interconnect matrix elements would be available to decouple the lateral response and specify stability characteristics. Therefore, setting $C_{y\delta_r} = 0$, a gain and control interconnect matrix was calculated that yielded the desired stability characteristics and also set the numerators of the β/δ_{a_p} and ϕ/δ_{r_p} transfer functions to zero. Results of this study (case 4) are illustrated in Tables 3 and 4. Table 4 shows that, in addition to making the β/β_{a_p} and ϕ/δ_{r_p} transfer function numerators vanish, pole-zero cancellation occurred in all transfer functions. In all aileron response transfer functions except β/δ_{a_p} (which was identically zero), the dutch roll mode was cancelled and in all rudder response transfer functions except ϕ/δ_{r_p} (which was identically zero) the spiral and roll mode roots were cancelled. These characteristics are altered insignificantly when the gain and control interconnect matrices calculated were used in the augmented vehicle with $C_{y\delta_r}$ included. It was thus determined that although exact decoupling while specifying all stability characteristics was not possible, for this example, a very nearly decoupled system could be obtained. The response of the vehicle to a 5° step aileron input δ_{a_p} is shown on Fig. 10. Also the vehicle response to a 5° step rudder input δ_{r_p} is illustrated in Fig. 11. These figures show that the β response has been practically eliminated from aileron input and that bank angle response has been practically eliminated from rudder input. An alternative approximate decoupling was studied retaining $C_{y\delta_r}$, specifying all the lateral stability characteristics, and further specifying that all numerator coefficients of the ϕ/δ_{r_p} and β/δ_{a_p} transfer functions be zero except the s^3 coefficient of the β/δ_{a_p} transfer function numerator. The resulting vehicle responses to step inputs δ_{a_p} and δ_{r_p} obtained through this procedure were indistinguishable from those on Figs. 10 and 11 and the calculated gain and interconnect differed only slightly from those calculated for case 4.

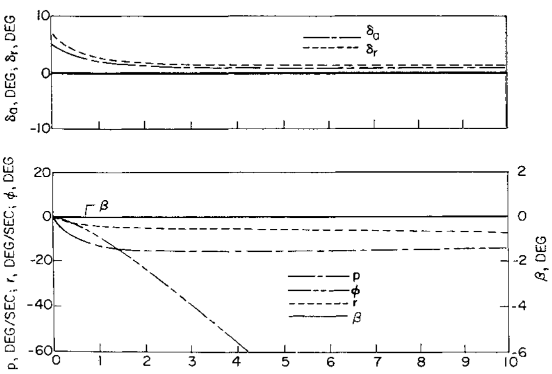

Fig. 10 Time history of the response of the augmented vehicle of case 4 to a 5° step aileron input.

Conclusions

A differential synthesis technique has been developed for multiple input linear feedback systems that allows direct computation of the set of feedback gains and control interconnects that yield arbitrarily selected flying qualities parameters. This was accomplished by parametrically relating differential gain and interconnect changes to differential changes in flying qualities parameters. The procedure has been illustrated by synthesizing a linear feedback control for the linearized lateral dynamics of a lifting-body entry vehicle with unacceptable flying qualities in the augmented condition.

At the present time, the computational method of synthesizing feedback controls presented in this paper is limited to systems of the type illustrated in Fig. 1. Extensions of this method to include dynamic elements in place of the constant F and G elements of Fig. 1 are areas of future research. In addition, analytic studies into establishing general methods of treating problems with singular gradient matrices are needed.

Table 4 Transfer functions for case 4

$d(s)$	=	$(s + 0.01)(s + 1.5)(s^2 + 0.9s + 9)$
p/δ_a	=	$-4.8(s^2 + 0.9s + 9)(s - 0.005)/d(s)$
ϕ/δ_a	=	$-5.1(s^2 + 0.9s + 9)/d(s)$
r/δ_a	=	$-1.3(s^2 + 0.9s + 9)(s + 0.07)/d(s)$
β/δ_a	=	0
p/δ_r	=	$+0.5(s + 0.01)(s + 1.5)(s + 0.09)/d(s)$
ϕ/δ_r	=	0
r/δ_r	=	$-1.9(s + 0.01)(s + 1.5)(s + 0.09)/d(s)$
β/δ_r	=	$+2.0(s + 0.01)(s + 1.5)/d(s)$

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Flight Test Evaluation of an Advanced Stability Augmentation System for B-52 Aircraft

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Results and comparisons with theoretical predictions are given for a flight test performance evaluation of an advanced stability augmentation system (SAS). The new SAS, developed for installation in the B-52G-H fleet, provides control of low-frequency structural modes as well as the conventional control of airplane rigid body motions. Flight test results are presented showing the SAS performance in terms of mode damping, fatigue damage rates, maximum expected stress, and ride quality for flight through turbulence. Comparisons are made between theoretically predicted and experimental results. The flight test results show significant reductions in dynamic response to turbulence with the advanced stability augmentation system. Reductions in response of the low-frequency antisymmetric structural modes and the Dutch roll mode were obtained with SAS. Lateral loads on the fin and aft fuselage during flight through turbulence were reduced by more than 20%. Fatigue damage rates due to turbulence were reduced more than 50% for these same structural locations. The flight control system configuration and test procedures used to evaluate the SAS performance are presented.

Introduction

An Air Force sponsored study was conducted by The Boeing Company during 1964 and 1965 to determine the changes to the B-52 flight control and stability augmentation systems that would provide meaningful improvements in the airplane structural life and in aerodynamic and structural stability in severe turbulence. This study was conducted as a part of a continuing program to provide B-52 fleet longevity and effectiveness to meet Air Force requirements during the next decade. The results of the study program, available in August 1965, indicated that significant reductions in structural fatigue and peak loads could be expected if an advanced stability augmentation system were installed on the B-52.

Development of the prototype stability augmentation system was accomplished during 1966 and 1967. Reference 1 summarizes SAS analyses and synthesis. Structural analyses conducted and a summary of the analytical results obtained are presented in Ref. 2. The system selected for development included both pitch and yaw stability augmentation.

A prototype model of the advanced SAS was designed, fabricated, and installed on a B-52H flight test airplane. Flight testing of the prototype SAS was completed in 1967 to optimize and demonstrate the SAS performance in terms of reducing peak structural loads and fatigue damage rates. The optimization was accomplished within the boundaries of adequate handling qualities and dynamic stability of the airplane.

The following sections present a general description of the flight control system configuration, the flight test approach, and results obtained. The flight test included flutter, SAS optimization, and performance testing. Performance testing included evaluation of handling qualities (Ref. 3) and dynamic response to atmospheric turbulence. General results obtained during gust response testing are described in this paper, including comparisons to analytical predictions.

Prototype SAS Configuration

A general description of the existing B-52 fleet flight configuration, which includes a yaw damper, is given in Ref. 2 along with study ground rules, SAS variations considered, and the SAS configuration selected for prototype flight testing. The two axis SAS consists of structural and rigid body motion sensors, and hydraulic actuators to position the elevator and rudder. These same hydraulic actuators also position the control surfaces on command from the primary flight control system and autopilot.

The yaw SAS functional configuration, illustrated in Fig. 1, utilizes a yaw rate gyro located at Body Station 695 (wing rear spar to body intersection) and a lateral accelerometer located at Body Station 1719 (stabilizer rear spar to body

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